



Improved Hardy inequalities and weighted Hardy type inequalities with spherical derivatives

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Abstract

The first purpose of this paper is to set up several enhancements of the Hardy type inequalities on certain subspaces of the Sobolev spaces. The second aim is to explore the role of the spherical derivative in such improvements and to prove some weighted versions of the Hardy inequality with spherical derivative. As applications of our results, we establish several Hardy's inequalities and the Hardy inequalities with spherical derivatives on half-spaces.

Keywords Hardy inequality · Spherical derivative · Best constant · Bessel pair

Mathematics Subject Classification 26D10 · 46E35 · 35A23

1 Introduction

Hardy's inequalities are one of the most used inequalities in analysis. They have been studied intensively and extensively in the literature due to their important roles in many areas of mathematics such as mathematical physics, spectral theory, analysis of linear and non-linear PDE, harmonic analysis and stochastic analysis. The interested

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reader is referred to the monographs [3,28,31,32,38,40], to name just a few, that are standard references on the subject.

In this note, we concern the Hardy inequality of the following form in $\mathbb{R}^N$, $N \geq 3$: Let u be in $D^{1,2}(\mathbb{R}^N)$, the completion with respect to $(\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{1}{2}}$ of $C_0^\infty(\mathbb{R}^N)$. Then $\frac{|u|^2}{|x|^2} \in L^1(\mathbb{R}^N)$. Moreover

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \quad (1)$$

The constant $\left(\frac{N-2}{2}\right)^2$ in (1) has been well-investigated in the literature. The fact that $\left(\frac{N-2}{2}\right)^2$ is optimal but is never achieved in (1) by nontrivial functions in $D^{1,2}(\mathbb{R}^N)$ is well-understood. Also several questions about the possible improvements of (1) have been raised. For instance, we may ask for the strengthened versions of the Hardy inequalities where extra nonnegative terms can be added to the RHS of (1). On the whole space $\mathbb{R}^N$, Ghoussoub and Moradifard showed in [28] that there is no strictly positive $V \in V^1((0, \infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \geq \int_{\mathbb{R}^N} V(|x|) u^2 dx$$

holds for all $u \in C_0^\infty(\mathbb{R}^N)$. However, the situation on bounded domain is different. Indeed, let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$. Then Brezis and Vázquez, in order to investigate the stability of singular solutions of certain nonlinear elliptic equations, proved in [8] that for all $u \in W_0^{1,2}(\Omega)$:

$$\int_{\Omega} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 dx \quad (2)$$

where ω_N is the volume of the unit ball and $z_0 = 2.4048\dots$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball, yet still not attained in $W_0^{1,2}(\Omega)$. Hence, they also conjectured in [8] that $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 dx$ is just a first term of an infinite series of extra terms that can be added to the RHS of (2). The results and questions of Brezis and Vázquez have been investigated by many authors. See [1,2,4,5,9,11–15,19,20,23,26,29,37,45,47], among others. In particular, in [25], Frank and Seiringer used the notion of the ground state representations to set up the following identity that improves, extends and unifies several results about the Hardy type inequalities:

$$\int_{\mathbb{R}^N} V(x) |\nabla u|^2 dx = \int_{\mathbb{R}^N} W(x) |u|^2 dx + \int_{\mathbb{R}^N} V(x) \varphi^2 \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2 dx.$$

Here φ is a positive weak solution of the Laplace equation

$$-\operatorname{div}(V(x) \nabla \varphi(x)) = W(x) \varphi(x).$$

See also the recent survey paper [10].

In order to study the Hardy type inequalities with radial weights, in [28], Ghous-soub and Moradifam introduced the notion of Bessel pairs and proved that if V and W are positive C^1 -functions on $(0, R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ and $\int_0^R r^{N-1} V(r) dr < \infty$ and $(r^{N-1} V, r^{N-1} W)$ is a Bessel pair on $(0, R)$, $0 < R \leq \infty$, then

$$\int_{B_R} V(|x|) |\nabla u|^2 dx \geq \int_{B_R} W(|x|) |u|^2 dx \quad (3)$$

for all $u \in C_0^\infty(B_R)$. Moreover, if (3) holds for all $u \in C_0^\infty(B_R)$, then $(r^{N-1} V, cr^{N-1} W)$ is a Bessel pair on $(0, R)$ for some $c > 0$. Here we say that a couple of C^1 -functions (V, W) is a Bessel pair on $(0, R)$ if the ordinary differential equation

$$(Vy')' + Wy = 0$$

has a positive solution on the interval $(0, R)$. See the book [28] for several examples and properties of Bessel pair. Actually, it was also showed in [18] that

$$\int_{B_R} V(|x|) |\nabla u|^2 dx = \int_{B_R} W(|x|) |u|^2 dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2 dx. \quad (4)$$

Another way to enhance the Hardy inequality that has received many attentions recently is to replace the usual gradient energy $\int_\Omega |\nabla u|^2 dx$ by smaller ones. For instance, we can use the quantity $\mathcal{R} := \frac{x}{|x|} \cdot \nabla$ instead of the usual ∇ . It can be noted that $\mathcal{R}u$ is the radial derivative of u . Indeed, in the polar coordinate, $|\mathcal{R}u| = |\partial_r u(r\sigma)|$

while $|\nabla u| = \left(|\partial_r u(r\sigma)|^2 + \frac{|\nabla_{\mathbb{S}^{N-1}} u(r\sigma)|^2}{r^2} \right)^{\frac{1}{2}}$. Actually, the radial derivation plays an important part in the literature. We just mention here the book [42] where the roles of the radial derivation $\mathcal{R}$ in the functional and geometric inequalities on homogeneous groups have been studied. We also note here that the Hardy type inequalities with radial gradient have been intensively investigated recently. We refer the reader to [16,17,27,30,33,34,36,39,41,43,46], for example.

The main motivation of this paper is the augmented versions of the Hardy inequality in [6,7,21]. More precisely, it was showed in [7,21] that when we consider the functions u on the orthogonal complement of the space of spherically symmetric functions, then the optimal constant of the Hardy inequality can be enhanced. Indeed, in [21], Ekholm and Frank established the following strengthened Hardy inequality and used

it to study the Lieb-Thirring inequalities for the operator $-\Delta - \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2} - V$ for certain potential V :

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx$$

for all $u \in C_0^\infty(\mathbb{R}^N)$ satisfying $\int_{\mathbb{S}^{N-1}} u(r\sigma) d\sigma = 0$ for all $r \geq 0$. This improved Hardy inequality has been investigated further in [6] where Bez, Machihara and Ozawa set up a version of the Hardy inequality with spherical derivative to explain for this phenomenon:

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx \geq (N-1) \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx$$

for all $u \in C_0^\infty(\mathbb{R}^N)$ satisfying $\int_{\mathbb{S}^{N-1}} u(r\sigma) d\sigma = 0$ for all $r \geq 0$.

Our first aim of this paper is to investigate the best constants of the Hardy inequalities on certain subspaces of $D^{1,2}(\mathbb{R}^N)$. More precisely, stimulated by the approach in [44], we will consider in this article the Hardy inequalities on the spaces $N_j(\Omega)$, $j \in \mathbb{N}$, that are defined as follows: Let $u \in C_0^\infty(\Omega)$. We then extend u as zero outside Ω and may consider that $u \in C_0^\infty(\mathbb{R}^N)$. Hence, we can decompose u into spherical harmonics as follows:

$$u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} f_k(r) \phi_k(\sigma),$$

where $\phi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace–Beltrami operator with corresponding eigenvalues $c_k = k(N+k-2)$, $k \geq 0$. We note that the corresponding components f_k are in $C_0^\infty(\Omega)$ and satisfy $f_k(r) = O(r^k)$, $f'_k(r) = O(r^{k-1})$ as $r \downarrow 0$. In particular

$$\phi_0(\sigma) = 1, c_0 = 0 \text{ and } f_0(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u ds.$$

Also, for any $k \in \mathbb{N}$:

$$\Delta u_k = \left(\Delta f_k(r) - c_k \frac{f_k(r)}{r^2} \right) \phi_k(\sigma).$$

For $j \in \mathbb{N}$, let

$$N_j(\Omega) = \{u \in C_0^\infty(\Omega) : u_0 = \dots = u_j = 0\} \\ = \left\{ u \in C_0^\infty(\Omega) : u = \sum_{k=j+1}^{\infty} u_k = \sum_{k=j+1}^{\infty} f_k(r) \phi_k(\sigma) \right\}.$$

Our first goal of this note is to show that the optimal constants of the Hardy type inequalities can indeed be improved on $N_j(\Omega)$:

Theorem 1 *Let $0 < R \leq \infty$, V and W be positive C^1 -functions on $0, R$ such that*

$$\int_{B_R} V(|x|) |\nabla u|^2 dx \geq \int_{B_R} W(|x|) |u|^2 dx$$

for all radial functions $u \in C_0^\infty(B_R \setminus \{0\})$. Then we have that for all $u \in C_0^\infty(B_R \setminus \{0\})$:

$$\begin{aligned} \int_{B_R} V(|x|) |\nabla u|^2 dx &\geq \int_{B_R} V(|x|) |\mathcal{R}u|^2 dx \\ &\geq \int_{B_R} W(|x|) |u|^2 dx \end{aligned} \quad (5)$$

and for all $u \in N_j(B_R \setminus \{0\})$:

$$\begin{aligned} &\int_{B_R} V(|x|) |\nabla u|^2 dx \\ &\geq \int_{B_R} \left[W(|x|) + (j+1)(N+j-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx. \end{aligned} \quad (6)$$

Note that the assumption in Theorem 1 is that the Hardy inequality holds for radial functions. Hence (5) can be considered as a symmetrization result of the Hardy inequality.

Using the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx,$$

we recover the improved Hardy inequality on $N_0(\mathbb{R}^N)$ in [7,21] as a direct consequence of our Theorem 1: for $u \in N_0(\mathbb{R}^N)$ (that is $\int_{\mathbb{S}^{N-1}} u(r\sigma) d\sigma = 0$ for all $r > 0$), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 dx &\geq \left[\left(\frac{N-2}{2} \right)^2 + (N-1) \right] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \\ &= \left(\frac{N}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \end{aligned} \quad (7)$$

Moreover, using the Hardy inequality in the spirit of Brezis and Vázquez:

$$\int_{B_R} |\nabla u|^2 dx - \left(\frac{N-2}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |B_R|^{-\frac{2}{N}} \int_{B_R} |u|^2 dx,$$

we obtain from our Theorem 1 that for $u \in N_0(B_R)$:

$$\int_{B_R} |\nabla u|^2 dx - \left(\frac{N}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |B_R|^{-\frac{2}{N}} \int_{B_R} |u|^2 dx. \quad (8)$$

More generally, using the weighted Hardy inequalities with Bessel pair (3), we get that if V and W are positive C^1 -functions on $(0, R)$ such that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$, $0 < R \leq \infty$, then for $u \in N_0(B_R \setminus \{0\})$:

$$\begin{aligned} &\int_{B_R} V(|x|) |\nabla u|^2 dx \\ &\geq \int_{B_R} \left[W(|x|) + (N-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx. \end{aligned} \quad (9)$$

Our next purpose is to provide a remainder for the enhanced Hardy inequality (9). More precisely, we will prove the following result:

Theorem 2 *Let $0 < R \leq \infty$, V and W be positive C^1 -functions on $(0, R)$. Assume that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$. Then we have that for all $u \in N_j(B_R \setminus \{0\})$:*

$$\begin{aligned} &\int_{B_R} V(|x|) |\nabla u(x)|^2 dx - \int_{B_R} \left[W(|x|) + (j+1)(N+j-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx \\ &\geq \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u(x)}{\varphi(|x|)} \right) \right|^2 dx. \end{aligned}$$

Here φ is the positive solution of

$$\left(r^{N-1} V(r) y'(r) \right)' + r^{N-1} W(r) y(r) = 0$$

on the interval $(0, R)$.

Using Theorem 2, we obtain the remainders of the Hardy inequalities (7) and (8) as follows:

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 dx - \left(\frac{N}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \\ & \geq \int_{\mathbb{R}^N} \frac{1}{|x|^{N-2}} \left| \mathcal{R} \left(|x|^{\frac{N-2}{2}} u(x) \right) \right|^2 dx \text{ for } u \in N_0(\mathbb{R}^N), \end{aligned}$$

and

$$\begin{aligned} & \int_{B_R} |\nabla u|^2 dx - \left(\frac{N}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx - z_0^2 \omega_N^{\frac{2}{N}} |B_R|^{-\frac{2}{N}} \int_{B_R} |u|^2 dx \\ & \geq \int_{B_R} \frac{J_0^2 \left(\frac{z_0}{R} |x| \right)}{|x|^{N-2}} \left| \mathcal{R} \left(\frac{|x|^{\frac{N-2}{2}} u(x)}{J_0 \left(\frac{z_0}{R} |x| \right)} \right) \right|^2 dx \text{ for } u \in N_0(B_R). \end{aligned}$$

Let $\mathcal{S}u = \nabla u - \left(\frac{x}{|x|} \cdot \nabla u \right) \frac{x}{|x|} = \nabla u - (\mathcal{R}u) \frac{x}{|x|}$. Since, $\mathcal{R}$ is the radial derivative, we can be considered $\mathcal{S}$ as the spherical derivative. Also, a direct computation yields

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathcal{S}u|^2 dx &= \int_{\mathbb{R}^N} \left| \nabla u - \frac{x}{|x|} \mathcal{R}u \right|^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx. \end{aligned}$$

Now, we note that on the Sobolev space $W^{1,2}(\mathbb{R}^N)$, the constant $\left(\frac{N-2}{2}\right)^2$ is sharp in both the Hardy inequalities

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx$$

and

$$\int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx.$$

Hence, we could say that on $W^{1,2}(\mathbb{R}^N)$, the spherical derivative $\mathcal{S}$ doesn't have any contribution to the sharpness of $\left(\frac{N-2}{2}\right)^2$ in the above Hardy inequalities. However,

the situation is very different on some certain subspaces of $W^{1,2}(\mathbb{R}^N)$. Indeed, on the subspace $N_0(\mathbb{R}^N)$ of $W^{1,2}(\mathbb{R}^N)$, although we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx,$$

it is not true that

$$\int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx \geq \left(\frac{N}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx.$$

In fact, we just have from Theorem 1 that

$$\int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx.$$

That means that in this case, $\int_{\mathbb{R}^N} |\mathcal{S}u|^2 dx$ plays the main role in the improvements of the optimal constants in the Hardy inequality on $N_0(\mathbb{R}^N)$. This observation has been verified in [6] where the authors proved that if $u \in N_0(\mathbb{R}^N)$ with

$$u = \sum_{k=1}^{\infty} f_k(r) \phi_k(\sigma),$$

then

$$\begin{aligned} \int_{B_R} |\mathcal{S}u|^2 dx &= (N-1) \int_{B_R} \frac{|u(x)|^2}{|x|^2} dx \\ &\quad + \sum_{k=2}^{\infty} (k-1)(N+k-1) \int_{B_R} \frac{|f_k(|x|)|^2}{|x|^2} dx \\ &\geq (N-1) \int_{B_R} \frac{|u(x)|^2}{|x|^2} dx. \end{aligned}$$

Our next goal of our paper is to show that this is also the case for the weighted Hardy type inequalities with Bessel pairs. More precisely, we will set up the following identities:

Theorem 3 *Let $0 < R \leq \infty$, V and W be positive C^1 -functions on $(0, R)$. Assume that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$. Then we have for all $u \in N_j(B_R \setminus \{0\})$ with*

$$u = \sum_{k=j+1}^{\infty} f_k(r) \phi_k(\sigma)$$

that

$$\begin{aligned} \int_{B_R} V(|x|) |\nabla u|^2 dx &= \int_{B_R} W(|x|) |u(x)|^2 dx \\ &+ \int_{B_R} V(|x|) |\mathcal{S}u|^2 dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u(x)}{\varphi(|x|)} \right) \right|^2 dx \end{aligned}$$

and

$$\begin{aligned} \int_{B_R} V(|x|) |\mathcal{S}u|^2 dx &= (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx \\ &+ \sum_{k=j+2}^{\infty} (k-j-1)(N+k+j-1) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx. \end{aligned}$$

Here φ is the positive solution of

$$\left(r^{N-1} V(r) y'(r) \right)' + r^{N-1} W(r) y(r) = 0$$

on the interval $(0, R)$.

As a consequence of our result, we get the following Hardy inequality with spherical derivative: for $u \in N_j(B_R)$:

$$\int_{B_R} |\mathcal{S}u|^2 dx \geq (j+1)(N+j-1) \int_{B_R} \frac{|u(x)|^2}{|x|^2} dx.$$

More generally, if $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$ then for all $u \in N_j(B_R \setminus \{0\})$:

$$\int_{B_R} V(|x|) |\mathcal{S}u|^2 dx \geq (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx.$$

An interesting phenomenon about the Hardy inequality is that the optimal constant $\left(\frac{N-2}{2}\right)^2$ can be improved on domains having 0 on their boundary. Indeed, it was showed that for domains with boundary singularity, the sharp constants of the Hardy inequality can be anywhere between $\left(\frac{N-2}{2}\right)^2$ and $\left(\frac{N}{2}\right)^2$. Also, if the best constant is

strictly less than $(\frac{N}{2})^2$, then it can be attained. See [22,24,28,35], to mention just a few. There exist domains such that the optimal constant is $(\frac{N}{2})^2$ and is not achieved, such as half-spaces $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$. For instance, in [5,35], the following inequality has been set up:

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 dx \geq \left(\frac{N}{2}\right)^2 \int_{\mathbb{R}_+^N} \frac{|u|^2}{|x|^2} dx.$$

Also, it was showed that $(\frac{N-2}{2})^2$ is the optimal constant in the following inequality (see [35,36], for instance):

$$\int_{\mathbb{R}_+^N} |\mathcal{R}u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}_+^N} \frac{|u|^2}{|x|^2} dx.$$

Hence again, the spherical derivative plays the crucial role in the improvement of the best constant of the Hardy inequalities on half-spaces $\mathbb{R}_+^N$.

Our last purpose of this paper is to point out a surprising relation between the space N_0 and the Sobolev space on half-space. More specifically, let $u \in C_0^\infty(B_R \cap \mathbb{R}_+^N)$. Then we define $v(x) = v(x', x_N) = u(x', |x_N|)$. A direct computation yields

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} v(r\sigma) d\sigma &= \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \geq 0\}} u(r\sigma', r\sigma_N) d\sigma + \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \leq 0\}} u(r\sigma', -r\sigma_N) d\sigma \\ &= \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \geq 0\}} u(r\sigma', r\sigma_N) d\sigma - \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \geq 0\}} u(r\sigma', r\sigma_N) d\sigma \\ &= 0. \end{aligned}$$

Hence $v \in N_0(B_R \setminus \{0\})$. Therefore, as applications of our main results, we can deduce several variants of the weighted Hardy inequalities on half-space using our results about the improved Hardy inequality on N_0 . We also obtain some versions of the Hardy inequality with spherical derivatives on $\mathbb{R}_+^N$. More precisely, we will prove the following results using our Theorem 1, Theorem 2 and Theorem 3:

Theorem 4 *Let $0 < R \leq \infty$, V and W be positive C^1 -functions on $(0, R)$. Assume that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$. Then we have for all $u \in C_0^\infty(B_R \cap \mathbb{R}_+^N)$:*

$$\int_{B_R \cap \mathbb{R}_+^N} V(|x|) |\nabla u|^2 dx \geq \int_{B_R \cap \mathbb{R}_+^N} \left[W(|x|) + (N-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx.$$

$$\int_{B_R \cap \mathbb{R}_+^N} V(|x|) |\mathcal{S}u|^2 dx \geq (N-1) \int_{B_R \cap \mathbb{R}_+^N} V(|x|) \frac{|u(x)|^2}{|x|^2} dx.$$

Our paper is organized as follows: In Sect. 2, we will provide a lemma that plays the important role in our proofs. Some consequences of our main results will also be deduced in Sect. 2. The proofs of main results will be given in Sect. 3.

2 A useful lemma and some consequences of main results

We state here a result that plays key role in our approach. Its proof can be found in [9, 16]. For the convenience of the reader, we will provide a proof here:

Lemma 1 *Assume that the decomposition of u into the spherical harmonics is $u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} f_k(r) \phi_k(\sigma)$ and assume that V is a nonnegative C^1 -function on $(0, \infty)$. Then we have*

$$\begin{aligned} \int_{\mathbb{R}^N} V(|x|) |u|^2 dx &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} V(|x|) |f_k(|x|)|^2 dx. \\ \int_{\mathbb{R}^N} V(|x|) |\mathcal{R}u|^2 dx &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} V(|x|) |\nabla f_k(|x|)|^2 dx \\ \int_{\mathbb{R}^N} V(|x|) |\nabla u|^2 dx &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} V(|x|) |\nabla f_k(|x|)|^2 + c_k V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx, \end{aligned}$$

Proof By polar coordinate and direct computations, we have

$$\begin{aligned} \int_{\mathbb{R}^N} V(|x|) |u|^2 dx &= \int_0^{\infty} \int_{S^{N-1}} V(r) \left| \sum_{k=0}^{\infty} f_k(r) \phi_k(\sigma) \right|^2 r^{N-1} dr d\sigma \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} V(|x|) |f_k(|x|)|^2 dx. \end{aligned}$$

Also

$$\begin{aligned} \int_{\mathbb{R}^N} V(|x|) |\mathcal{R}u|^2 dx &= \int_{\mathbb{R}^N} V(|x|) \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx \\ &= \int_0^{\infty} \int_{S^{N-1}} V(r) \left| \sum_{k=0}^{\infty} \frac{\partial u_k}{\partial r} \right|^2 r^{N-1} dr d\sigma \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \int_{S^{N-1}} V(r) \left| \sum_{k=0}^\infty f'_k(r) \phi_k(\sigma) \right|^2 r^{N-1} dr d\sigma \\
&= \sum_{k=0}^\infty \left| S^{N-1} \right| \int_0^\infty V(r) |f'_k(r)|^2 r^{N-1} dr \\
&= \sum_{k=0}^\infty \int_{\mathbb{R}^N} V(|x|) |\nabla f_k(|x|)|^2 dx.
\end{aligned}$$

Next, we note that

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(|x|) u \Delta u dx \\
&= \int_0^\infty \int_{S^{N-1}} V(r) \left(\sum_{k=0}^\infty f_k(r) \phi_k(\sigma) \right) \\
&\quad \times \left(\sum_{k=0}^\infty \left(f''_k(r) + \frac{N-1}{r} f'_k(r) - c_k \frac{f_k(r)}{r^2} \right) \phi_k(\sigma) \right) r^{N-1} dr d\sigma \\
&= \sum_{k=0}^\infty \left| S^{N-1} \right| \int_0^\infty V(r) f_k(r) \left(f''_k(r) + \frac{N-1}{r} f'_k(r) - c_k \frac{f_k(r)}{r^2} \right) r^{N-1} dr d\sigma
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} u \nabla V(|x|) \cdot \nabla u dx \\
&= \int_0^\infty \int_{S^{N-1}} \left(\sum_{k=0}^\infty f_k(r) \phi_k(\sigma) \right) \\
&\quad \times \left[\frac{V'(r)}{r} (r, \sigma) \cdot \sum_{k=0}^\infty \left(f'_k(r) \phi_k(\sigma), \frac{f_k(r)}{r} \nabla_{S^{N-1}} \phi_k(\sigma) \right) \right] r^{N-1} dr d\sigma \\
&= \sum_{k=0}^\infty \left| S^{N-1} \right| \int_0^\infty V'(r) f_k(r) f'_k(r) r^{N-1} dr d\sigma.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} V(|x|) |\nabla u|^2 dx = - \int_{\mathbb{R}^N} u \nabla \cdot (V(|x|) \nabla u) dx$$

$$\begin{aligned}
 &= - \int_{\mathbb{R}^N} u V(|x|) \Delta u dx - \int_{\mathbb{R}^N} u \nabla V(|x|) \cdot \nabla u dx \\
 &= - \sum_{k=0}^{\infty} |S^{N-1}| \int_0^{\infty} \left[V(r) f_k(r) \left(f_k''(r) + \frac{N-1}{r} f_k'(r) - c_k \frac{f_k(r)}{r^2} \right) \right. \\
 &\quad \left. + V'(r) f_k(r) f_k'(r) \right] r^{N-1} dr \\
 &= \sum_{k=0}^{\infty} |S^{N-1}| \int_0^{\infty} \left[V(r) |f_k'(r)|^2 + c_k V(r) \frac{f_k(r)}{r^2} \right] r^{N-1} dr \\
 &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} V(|x|) |\nabla f_k(|x|)|^2 + c_k V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.
 \end{aligned}$$

□

2.1 Some consequences of main results

In this subsection, we list a few augmented Hardy type inequalities as simple consequences of our main results.

First, we can check that $(r^{\alpha+1}, (\frac{\alpha}{2})^2 r^{\alpha-1})$ is a Bessel pair on $(0, \infty)$ with $\varphi = r^{-\frac{\alpha}{2}}$. Hence we get the following Hardy type inequalities for $u \in N_j(\mathbb{R}^N \setminus \{0\})$:

$$\begin{aligned}
 \int_{\mathbb{R}^N} |x|^{\alpha-N+2} |\nabla u|^2 &\geq \left[\left(\frac{\alpha}{2} \right)^2 + (j+1)(N+j-1) \right] \int_{\mathbb{R}^N} |x|^{\alpha-N} |u(x)|^2 dx \\
 \int_{\mathbb{R}^N} |x|^{\alpha-N+2} |\mathcal{S}u|^2 &\geq (j+1)(N+j-1) \int_{\mathbb{R}^N} |x|^{\alpha-N} |u(x)|^2 dx.
 \end{aligned}$$

We also obtain following Hardy type inequalities on $\mathbb{R}_+^N$ for $u \in C_0^\infty(\mathbb{R}_+^N)$:

$$\begin{aligned}
 \int_{\mathbb{R}_+^N} |x|^{\alpha-N+2} |\nabla u|^2 &\geq \left[\left(\frac{\alpha}{2} \right)^2 + N-1 \right] \int_{\mathbb{R}_+^N} |x|^{\alpha-N} |u(x)|^2 dx \\
 \int_{\mathbb{R}_+^N} |x|^{\alpha-N+2} |\mathcal{S}u|^2 &\geq (N-1) \int_{\mathbb{R}_+^N} |x|^{\alpha-N} |u(x)|^2 dx.
 \end{aligned}$$

Next, for $R > 0$, we note that $\left(r^{\alpha+1}, \left(\frac{\alpha}{2}\right)^2 r^{\alpha-1} + \frac{z_0^2}{R^2} r^{\alpha+1}\right)$ is a Bessel pair on $(0, R)$ with $\varphi = r^{-\frac{\alpha}{2}} J_0\left(\frac{r z_0}{R}\right) = r^{-\frac{\alpha}{2}} J_{0;R}(r)$. Hence we get the following Hardy type inequality for $u \in N_j(B_R \setminus \{0\})$:

$$\begin{aligned} & \int_{B_R} |x|^{\alpha-N+2} |\nabla u|^2 \\ & \geq \left[\left(\frac{\alpha}{2}\right)^2 + (j+1)(N+j-1) \right] \int_{B_R} |x|^{\alpha-N} |u(x)|^2 dx + \frac{z_0^2}{R^2} \int_{B_R} |x|^{\alpha-N+2} |u(x)|^2 dx. \end{aligned}$$

Similarly, we also get that for $u \in C_0^\infty(B_R \cap \mathbb{R}_+^N)$:

$$\begin{aligned} & \int_{B_R \cap \mathbb{R}_+^N} |x|^{\alpha-N+2} |\nabla u|^2 \\ & \geq \left[\left(\frac{\alpha}{2}\right)^2 + N - 1 \right] \int_{B_R \cap \mathbb{R}_+^N} |x|^{\alpha-N} |u(x)|^2 dx + \frac{z_0^2}{R^2} \int_{B_R \cap \mathbb{R}_+^N} |x|^{\alpha-N+2} |u(x)|^2 dx. \end{aligned}$$

It is worthy to mention that we can deduce as many improved Hardy's inequalities on N_j and on half-spaces as we can form Bessel pairs. The interested reader is referred to the book [28] where several examples of Bessel pairs are provided.

3 Proofs of main results

Proof of Theorem 1 Assume that the decomposition of u into the spherical harmonics is $u = \sum_{k=0}^\infty u_k = \sum_{k=0}^\infty f_k(r) \phi_k(\sigma)$. By assumption, we have

$$\int_{\mathbb{R}^N} V(|x|) |\nabla f_k(|x|)|^2 dx \geq \int_{B_R} W(|x|) |f_k(|x|)|^2 dx.$$

Hence by Lemma 1, we have

$$\begin{aligned} \int_{\mathbb{R}^N} V(|x|) \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx &= \sum_{k=0}^\infty \int_{\mathbb{R}^N} V(|x|) |\nabla f_k(|x|)|^2 dx \\ &\geq \sum_{k=0}^\infty \int_{B_R} W(|x|) |f_k(|x|)|^2 dx \\ &= \int_{\mathbb{R}^N} W(|x|) |u|^2 dx. \end{aligned}$$

Now, assume that $u \in N_j$. Then we can decompose u into spherical harmonics as follows:

$$u = \sum_{k=j+1}^{\infty} u_k = \sum_{k=j+1}^{\infty} f_k(r) \phi_k(\sigma).$$

By Lemma 1, we get

$$\int_{B_R} V(|x|) |\nabla u|^2 dx = \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) |\nabla f_k(|x|)|^2 + c_k V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.$$

We note that by the assumption

$$\int_{B_R} V(|x|) |\nabla f_k(|x|)|^2 dx \geq \int_{B_R} W(|x|) |f_k(|x|)|^2 dx.$$

Also, for $k \geq j+1$:

$$c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \geq (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.$$

Hence

$$\begin{aligned} \int_{B_R} V(|x|) |\nabla u|^2 dx &\geq \sum_{k=j+1}^{\infty} \int_{B_R} W(|x|) |f_k(|x|)|^2 dx \\ &\quad + \sum_{k=j+1}^{\infty} (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \\ &= \int_{B_R} W(|x|) |u(x)|^2 dx + (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx \\ &= \int_{B_R} \left[W(|x|) + (j+1)(N+j-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx. \end{aligned}$$

□

Proof of Theorem 2 Assume that we can decompose u into spherical harmonics as follows:

$$u = \sum_{k=j+1}^{\infty} u_k = \sum_{k=j+1}^{\infty} f_k(r) \phi_k(\sigma).$$

We have from Lemma 1 that

$$\int_{B_R} V(|x|) |\nabla u|^2 dx = \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) |\nabla f_k(|x|)|^2 dx + c_k V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.$$

We note that by the identity (4), we get

$$\int_{B_R} V(|x|) |\nabla f_k(|x|)|^2 dx = \int_{B_R} W(|x|) |f_k(|x|)|^2 dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{f_k(|x|)}{\varphi(|x|)} \right) \right|^2 dx.$$

Also, for $k \geq j+1$:

$$c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \geq (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.$$

Hence

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 dx \\ & \geq \sum_{k=j+1}^{\infty} \int_{B_R} W(|x|) |f_k(|x|)|^2 dx + \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{f_k(|x|)}{\varphi(|x|)} \right) \right|^2 dx \\ & \quad + \sum_{k=j+1}^{\infty} (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx. \end{aligned}$$

Now by Lemma 1, we obtain

$$\begin{aligned} & \sum_{k=j+1}^{\infty} \int_{B_R} W(|x|) |f_k(|x|)|^2 dx = \int_{B_R} W(|x|) |u(x)|^2 dx \\ & \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx = \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx. \end{aligned}$$

Also by noting that

$$\frac{u}{\varphi} = \sum_{k=j+1}^{\infty} \frac{f_k(r)}{\varphi(r)} \phi_k(\sigma),$$

we get from Lemma 1 that

$$\sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{f_k(|x|)}{\varphi(|x|)} \right) \right|^2 dx = \int_{\mathbb{R}^N} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u}{\varphi(|x|)} \right) \right|^2 dx.$$

Hence

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 dx \\ & \geq \int_{B_R} W(|x|) |u(x)|^2 dx + (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx \\ & \quad + \int_{\mathbb{R}^N} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u}{\varphi(|x|)} \right) \right|^2 dx. \end{aligned}$$

Equivalently,

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 dx - \int_{B_R} \left[W(|x|) + (j+1)(N+j-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx \\ & \geq \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u(x)}{\varphi(|x|)} \right) \right|^2 dx. \end{aligned}$$

□

Proof of Theorem 3 Assume that we can decompose u into spherical harmonics as follows:

$$u = \sum_{k=j+1}^{\infty} u_k = \sum_{k=j+1}^{\infty} f_k(r) \phi_k(\sigma).$$

We have from Lemma 1 that

$$\int_{B_R} V(|x|) |\nabla u|^2 dx = \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) |\nabla f_k(|x|)|^2 + c_k V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.$$

We note again that

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla f_k(|x|)|^2 dx \\ &= \int_{B_R} W(|x|) |f_k(|x|)|^2 dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{f_k(|x|)}{\varphi(|x|)} \right) \right|^2 dx. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 dx \\ &= \sum_{k=j+1}^{\infty} \int_{B_R} W(|x|) |f_k(|x|)|^2 dx + \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{f_k(|x|)}{\varphi(|x|)} \right) \right|^2 dx \\ &+ \sum_{k=j+1}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \\ &= \sum_{k=j+1}^{\infty} \int_{B_R} W(|x|) |f_k(|x|)|^2 dx + \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{f_k(|x|)}{\varphi(|x|)} \right) \right|^2 dx \\ &+ \sum_{k=j+1}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx. \end{aligned}$$

Using the fact that

$$\frac{u}{\varphi} = \sum_{k=j+1}^{\infty} \frac{f_k(r)}{\varphi(r)} \phi_k(\sigma)$$

and applying Lemma 1, we get

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 dx - \int_{B_R} W(|x|) |u(x)|^2 dx \\ &= \sum_{k=j+1}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u(x)}{\varphi(|x|)} \right) \right|^2 dx. \end{aligned}$$

Now, we recall the following identity in [17]:

$$\int_{B_R} V(|x|) |\mathcal{R}u|^2 dx - \int_{B_R} W(|x|) |u(x)|^2 dx$$

$$= \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u(x)}{\varphi(|x|)} \right) \right|^2 dx.$$

As a consequence

$$\begin{aligned} & \int_{B_R} V(|x|) |\mathcal{S}u|^2 dx \\ &= \int_{B_R} V(|x|) |\nabla u|^2 dx - \int_{B_R} V(|x|) |\mathcal{R}u|^2 dx \\ &= \sum_{k=j+1}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \\ &= (j+1)(N+j-1) \sum_{k=j+1}^{\infty} \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \\ &\quad + \sum_{k=j+1}^{\infty} (c_k - (j+1)(N+j-1)) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \\ &= (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx \\ &\quad + \sum_{k=j+1}^{\infty} (k-j-1)(N+k+j-1) \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx. \end{aligned}$$

Hence, we get

$$\begin{aligned} & \int_{B_R} V(|x|) |\nabla u|^2 dx \\ &= \int_{B_R} W(|x|) |u(x)|^2 dx + \int_{B_R} V(|x|) |\mathcal{S}u|^2 dx \\ &\quad + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u(x)}{\varphi(|x|)} \right) \right|^2 dx \end{aligned}$$

and

$$\int_{B_R} V(|x|) |\mathcal{S}u|^2 dx$$

$$\begin{aligned}
 &= (j+1)(N+j-1) \int_{B_R} V(|x|) \frac{|u(x)|^2}{|x|^2} dx \\
 &+ \sum_{k=j+1}^{\infty} (k-j-1)(N+k+j-1) \int_{B_R} V(|x|) \frac{|f_k(x)|^2}{|x|^2} dx.
 \end{aligned}$$

□

Proof of Theorem 4 Let $u \in C_0^\infty(B_R \cap \mathbb{R}_+^N)$. We define $v(x) = v(x', x_N) = u(x', |x_N|)$. Then

$$\begin{aligned}
 \int_{\mathbb{S}^{N-1}} v(r\sigma) d\sigma &= \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \geq 0\}} u(r\sigma', r\sigma_N) d\sigma + \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \leq 0\}} u(r\sigma', -r\sigma_N) d\sigma \\
 &= \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \geq 0\}} u(r\sigma', r\sigma_N) d\sigma - \int_{\mathbb{S}^{N-1} \cap \{\sigma_N \geq 0\}} u(r\sigma', r\sigma_N) d\sigma = 0.
 \end{aligned}$$

Hence $v \in N_0(B_R \setminus \{0\})$. As a consequence, by Theorem 1 and Theorem 2, if $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$, then

$$\begin{aligned}
 &\int_{B_R} V(|x|) |\nabla v|^2 dx \\
 &\geq \int_{B_R} \left[W(|x|) + (N-1) \frac{V(|x|)}{|x|^2} \right] |v(x)|^2 dx.
 \end{aligned}$$

On the other hand,

$$\int_{B_R} V(|x|) |\nabla v|^2 dx = 2 \int_{B_R \cap \mathbb{R}_+^N} V(|x|) |\nabla u|^2 dx$$

and

$$\begin{aligned}
 &\int_{B_R} \left[W(|x|) + (N-1) \frac{V(|x|)}{|x|^2} \right] |v(x)|^2 dx \\
 &= 2 \int_{B_R \cap \mathbb{R}_+^N} \left[W(|x|) + (N-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx.
 \end{aligned}$$

Hence, we deduce

$$\int_{B_R \cap \mathbb{R}_+^N} V(|x|) |\nabla u|^2 dx \geq \int_{B_R \cap \mathbb{R}_+^N} \left[W(|x|) + (N-1) \frac{V(|x|)}{|x|^2} \right] |u(x)|^2 dx.$$

Now, by Theorem 3, we obtain

$$\int_{B_R} V(|x|) |\mathcal{S}v|^2 dx \geq (N-1) \int_{B_R} V(|x|) \frac{|v(x)|^2}{|x|^2} dx.$$

A direct computation yields

$$\int_{\tilde{B}_R} V(|x|) |\mathcal{S}v|^2 dx = 2 \int_{B_R \cap \mathbb{R}_+^N} V(|x|) |\mathcal{S}v|^2 dx.$$

Hence, we deduce

$$\int_{B_R \cap \mathbb{R}_+^N} V(|x|) |\mathcal{S}u|^2 dx \geq (N-1) \int_{B_R \cap \mathbb{R}_+^N} V(|x|) \frac{|u(x)|^2}{|x|^2} dx.$$

□

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